

Magnus and Dyson Series for Master Integrals

Stefano Di Vita

based on work with M. Argeri, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi, V. Yundin

Max-Planck-Institut für Physik, München

LoopFest XIII, New York City College of Technology
June 20th, 2014



Outline

- 1 Differential Equations and Feynman integrals
- 2 Magnus and Dyson series
- 3 Some applications

Outline

1 Differential Equations and Feynman integrals

2 Magnus and Dyson series

3 Some applications

Motivation (?)



Someone would probably like to cross check his result ...

IBP reduction of Feynman integrals

[Chetyrkin and Tkachov; Laporta]

- goal: $I_d(\vec{s}, \vec{m}; \vec{n}) = \int \prod \{d^d k_i\} \frac{1}{D_1^{n_1} \cdots D_p^{n_p}}$ (\vec{s} = ext. kin. invariants, \vec{m} = int. masses)
- IBP: $\int \prod \{d^d k_i\} \frac{\partial}{\partial k_j^\mu} \left[\frac{v^\mu}{D_1^{n_1} \cdots D_p^{n_p}} \right] = 0$, v^μ = any loop or ext. mom.
- Lorentz inv: $\sum_a \left[p_a^\nu \frac{\partial}{\partial p_a^\mu} - p_a^\mu \frac{\partial}{\partial p_a^\nu} \right] I_d(\vec{s}, \vec{m}; \vec{n}) = 0$
- By recursive application, $I_d(\vec{s}, \vec{m}; \vec{n}) \rightarrow \sum_i c_d^{(i)}(\vec{m}; \vec{n}) M_d^{(i)}(\vec{m}, \vec{s})$
- $c_d^{(i)}(\vec{s}, \vec{m}; \vec{n})$ = rational functions of polynomials
- $M_d^{(i)}(\vec{s}, \vec{m})$ = **few** Master Integrals encoding analytic structure
- Public computer codes AIR [Anastasiou, Lazopoulos 04], FIRE [AV Smirnov 08], REDUCE [Studerus 10; Studerus, von Manteuffel 12], LiteRed [Lee 12]

- ✓ rational coefficients \leftrightarrow purely algebraic effort, CAS, patience
- ✗ MI's \leftrightarrow 1-loop, n-point was too easy, unsolved in the general case

MI's evaluation: choose your weapon wisely . . .



- ▶ Feynman/Schwinger parameter representation
- ▶ Mellin-Barnes representation
- ▶ Sector Decomposition
- ▶ Differential Equations
- ▶ ... or pick your favorite outsider

My old approach

Recycling known results, thus avoiding explicit integrations :)

My new approach

Integrating differential eq's **only if brought in canonical form!** [Henn 13]

Differential Equations for Feynman integrals (1)

[Kotikov; Remiddi; Caffo, Czyz, Remiddi; Gehrmann, Remiddi]

The MI's are functions of

- ▶ the kinematic invariants \vec{S} built with the external momenta and masses
- ▶ the internal masses
- ▶ the number of spacetime dimensions d

The previously introduced relations force them to obey

Linear systems of 1st order DE, for arbitrary d

- ▶ if possible (e.g. 1-loop bubble) can be integrated directly
- ▶ otherwise can be Laurent-expanded around some d_0
⇒ system of **chained** eq's for the expansion coefficients
- ▶ in any case, need to impose a suitable **boundary condition**
⇒ exploit regularity at pseudo-thresh. or known kin. limits

Differential Equations for Feynman integrals (2)

► 2pts

$$p^2 \frac{\partial}{\partial p^2} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu}$$

► 3pts

$$P^2 \frac{\partial}{\partial P^2} = \left[A \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left(p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right]$$

$P = p_1 + p_2$ and A, B rational coefficients

► 4pts

$$P^2 \frac{\partial}{\partial P^2} = \left[C \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E(p_{1,\mu} + p_{3,\mu}) \left(\frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right]$$

$$Q^2 \frac{\partial}{\partial Q^2} = \left[F \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + G p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + H(p_{2,\mu} - p_{1,\mu}) \left(\frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} + \frac{\partial}{\partial p_{3,\mu}} \right) \right]$$

$P = p_1 + p_2$, $Q = p_1 - p_3$ and C, D, E, F, G, H rational coefficients

shamelessly stolen from [Argeri, Mastrolia 07]

IBP/LI id's \Rightarrow rhs can be expressed as a linear combination of *the same MI's* we are taking derivatives of, we obtain a **closed system!**

Differential Equations for Feynman integrals (3)

- ▶ typically the MI's \vec{f} are identified after application of Laporta's alg.
- ▶ nevertheless, the choice of the MI's is **not unique**
- ▶ can an optimal set be chosen? [Henn 13]
- ▶ let us define $\epsilon = (d_0 - d)/2$ (typically $d_0 = 4$)
- ▶ choose suitable dimensionless variables \vec{x}
- ▶ define a linear basis change $\vec{f} = B \vec{g}$

bad choice

$$\partial_{x_i} g(\vec{x}, \epsilon) = A_{x_i}(\vec{x}, \epsilon) g(\vec{x}, \epsilon)$$

good choice (if \exists)

$$\partial_{x_i} g(\vec{x}, \epsilon) = \epsilon A_{x_i}(\vec{x}) g(\vec{x}, \epsilon)$$

- ▶ If the MI's can be expressed in terms of generalized polylogs,
 $A_{x_i} = \sum_k \frac{a_{x_i}^{(k)}}{x_i - x_i^{(k)}}$, where $a_{x_i}^{(k)}$ are *rational numbers*
- ▶ expanding $\vec{g} = \sum_n \epsilon^n \vec{g}^{(n)}$, the integration trivializes

How to choose a good basis?

Many successful applications

[Henn 13; Henn and Smirnov 13; Henn, Smirnov and Smirnov 13; Henn, Melnikov and Smirnov 14; Caron-Huot and Henn 14]

Problems

- ▶ does a “good” basis exist in general? (open math problem)
- ▶ devising an algorithm for finding the good B is a formidable task

Some criteria for identifying good MI's [Henn 13]

- ▶ obey canonical DE :)
- ▶ $D \rightarrow \delta$, check if UT/rescale/combine
- ▶ inspect Feynman par. representation ($1/[\dots]^{n+\epsilon} \sim 1/[\dots]^n$)

Our proposal [Argeri, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, DV 14]

if MI's obey a DE system linear in ϵ , then $\exists B$ which brings the DE in canonical form

Outline

1 Differential Equations and Feynman integrals

2 Magnus and Dyson series

3 Some applications

A convenient tool: Magnus series expansion

[Magnus 54]

- ▶ a generic matrix linear system of 1st order ODE

$$\partial_x Y(x) = A(x)Y(x), \quad Y(x_0) = Y_0$$

- ▶ in the general non-commutative case, Magnus theorem tells us that

$$Y(x) = e^{\Omega(x, x_0)} Y(x_0) \equiv e^{\Omega(x)} Y_0$$

- ▶ with $\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x)$ and

$$\Omega_1(x) = \int_{x_0}^x d\tau_1 A(\tau_1),$$

$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^x d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)]$$

$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^t d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]]$$

Relation with Dyson series

[Blanes, Casas, Oteo and Ros 09]

Magnus \leftrightarrow Dyson series. Dyson expansion of the solution Y in terms of the *time-ordered* integrals Y_n

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x)$$

$$Y_n(x) \equiv \int_{x_0}^x d\tau_1 \dots \int_{x_0}^{\tau_{n-1}} d\tau_n A(\tau_1)A(\tau_2)\cdots A(\tau_n),$$

Then

$$Y(x) = e^{\Omega(x)} Y_0 \quad \Rightarrow \quad \sum_{j=1}^{\infty} \Omega_j(x) = \log \left(Y_0 + \sum_{n=1}^{\infty} Y_n(x) \right)$$

and

$$Y_1 = \Omega_1,$$

$$Y_2 = \Omega_2 + \frac{1}{2!} \Omega_1^2,$$

$$Y_3 = \Omega_3 + \frac{1}{2!} (\Omega_1 \Omega_2 + \Omega_2 \Omega_1) + \frac{1}{3!} \Omega_1^3$$

Integrating ϵ -linear DE's [Argeri, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, DV 14]

- 1 start with DE linear in ϵ (may need a bit of trial and error + expertise)

$$\partial_x f(\epsilon, x) = A(\epsilon, x)f(\epsilon, x), \quad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)$$

- 2 basis change with Magnus: $f(\epsilon, x) = B_0(x) g(\epsilon, x)$

$$B_0(x) \equiv e^{\Omega[A_0](x, x_0)} \quad \leftrightarrow \quad \partial_x B_0(x) = A_0(x)B_0(x)$$

- 3 obtain a canonical system for the g 's

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x)g(\epsilon, x), \quad \hat{A}_1(x) = B_0^{-1}(x)A_1(x)B_0(x)$$

- 4 obtain the solution with Magnus (or Dyson)

$$g(\epsilon, x) = B_1(\epsilon, x)g_0(\epsilon), \quad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x, x_0)}$$

- 5 ϵ -expansion of g 's will have uniform weight ("transcendentality")
(if $g(0)$'s are chosen wisely)

Outline

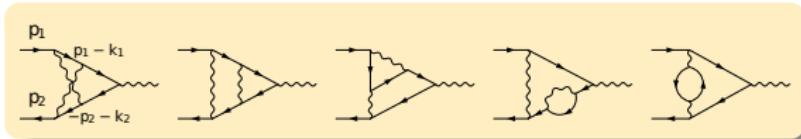
1 Differential Equations and Feynman integrals

2 Magnus and Dyson series

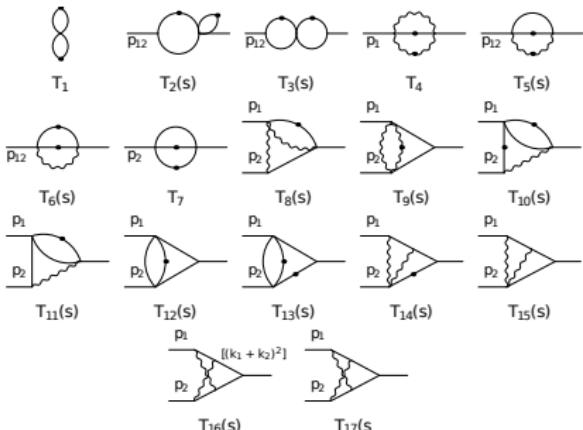
3 Some applications

(1) Two-loop QED vertices [Bonciani, Mastrolia and Remiddi 03]

17 MI's for all relevant topologies



$$-\frac{s}{m^2} = \frac{(1-x)^2}{x}$$



The f 's obey an ϵ -linear DE [ADVMMMSST '14]

$$\begin{aligned}
f_1 &= \epsilon^2 T_1 & f_2 &= \epsilon^2 T_2 & f_3 &= \epsilon^2 T_3 & f_4 &= \epsilon^2 T_4 & f_5 &= \epsilon^2 T_5 \\
f_6 &= \epsilon^2 T_6 & f_7 &= \epsilon^2 T_7 & f_8 &= \epsilon^3 T_8 & f_9 &= \epsilon^3 T_9 & f_{10} &= \epsilon^2 T_{10} \\
f_{11} &= \epsilon^3 T_{11} & f_{12} &= \epsilon^3 T_{12} & f_{13} &= \epsilon^2 T_{13} & f_{14} &= \epsilon^3 T_{14} & f_{15} &= \epsilon^4 T_{15} \\
f_{16} &= \epsilon^4 T_{16} & f_{17} &= \epsilon^4 T_{17}
\end{aligned}$$

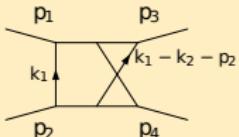
After getting rid of A_0 , the g 's obey a canonical DE

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \quad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1+x} + \frac{M_3}{1-x}$$

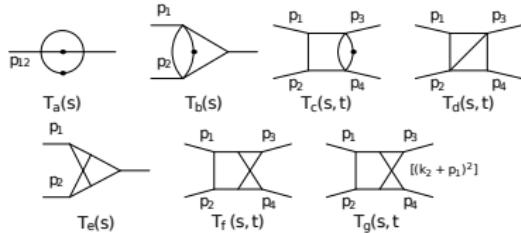
(2) Two-loop non-planar box

[Tausk 99; Anastasiou, Gehrmann, Oleari, Remiddi, Tausk 00]

12 MI's for the crossed topology



$$x = -\frac{t}{s}, \quad s > 0, t < 0, |s| > |t|$$



The f 's obey an ϵ -linear DE [ADVMMST '14]

$$f_1 = \epsilon^2 s T_a(s), \quad f_2 = \epsilon^2 t T_a(t), \quad f_3 = \epsilon^2 u T_a(u),$$

$$f_4 = \epsilon^3 s T_b(s), \quad f_5 = \epsilon^3 s t T_c(s, t), \quad f_6 = \epsilon^3 s u T_c(s, u),$$

$$f_7 = \epsilon^4 u T_d(s, t), \quad f_8 = \epsilon^4 s T_d(t, u), \quad f_9 = \epsilon^4 t T_d(u, s),$$

$$f_{10} = \epsilon^4 s^2 T_e(s),$$

$$f_{11} = \epsilon^4 s t u T_f(s, t) - \frac{3}{4s(4\epsilon+1)} \left[\epsilon^2 \left(s^2 T_a(s) + t^2 T_a(t) + u^2 T_a(u) \right) - 4\epsilon^4 \left(u^2 T_d(s, t) + s^2 T_d(t, u) + t^2 T_d(u, s) \right) \right],$$

$$f_{12} = \epsilon^4 s t T_g(s, t) - \frac{3}{8u(4\epsilon+1)} \left[\epsilon^2 \left(s^2 T_a(s) + t^2 T_a(t) + u^2 T_a(u) \right) - 4\epsilon^4 \left(u^2 T_d(s, t) + s^2 T_d(t, u) + t^2 T_d(u, s) \right) \right],$$

After getting rid of A_0 , the g 's obey a canonical DE

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \quad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1-x}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(3) Higgs + 1Jet

[Mastrolia, Schubert, Yundin, DV... work in progress!]



Higgs



Jet

(3) Higgs + 1 Jet

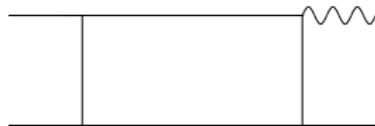
[Mastrolia, Schubert, Yundin, DV... work in progress!]



Higgs



Jet



4 MI's [Kindergarten]

(3) Higgs + 1 Jet

[Mastrolia, Schubert, Yundin, DV... work in progress!]



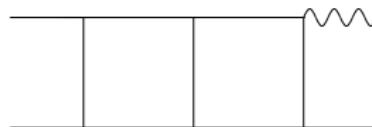
Higgs



Jet



4 MI's [Kindergarten]



18 MI's [Gehrmann+Remiddi '00]

(3) Higgs + 1 Jet

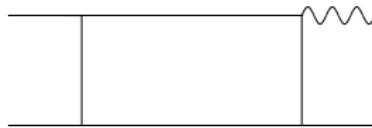
[Mastrolia, Schubert, Yundin, DV... work in progress!]



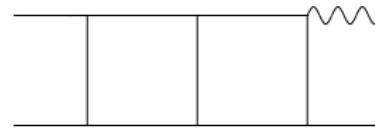
Higgs



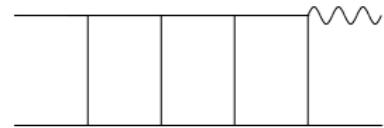
Jet



4 MI's [Kindergarten]



18 MI's [Gehrmann, Remiddi 00]



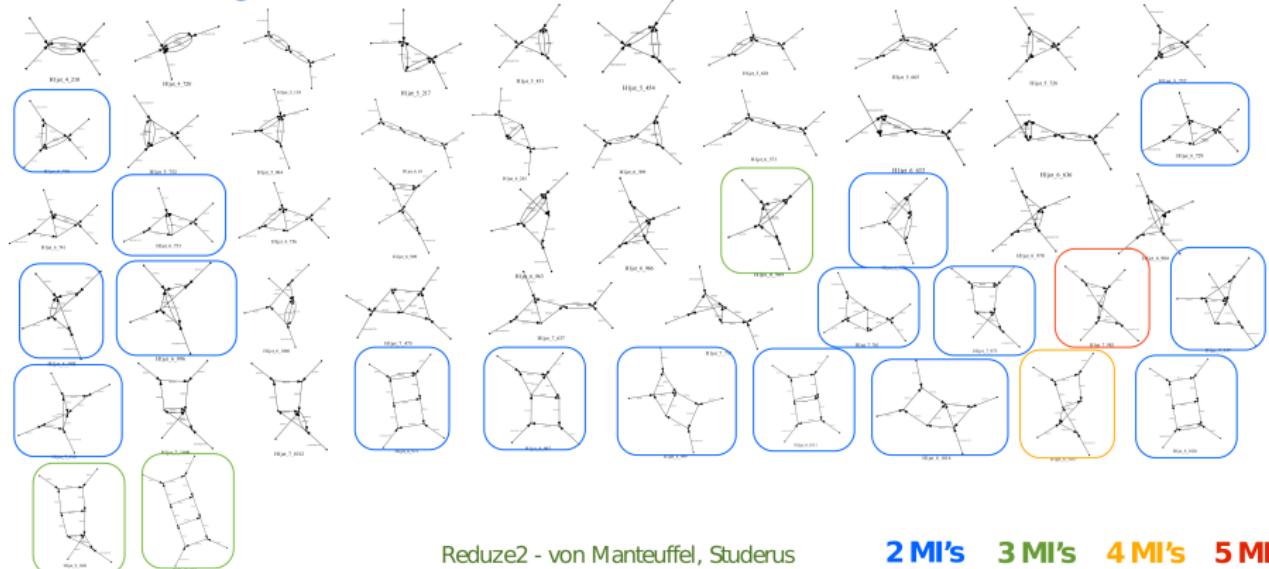
85 MI's!!

our own GHPL code + REDUZE [Studerus, Studerus and von Manteuffel] + GiNaC [Bauer, Frink and Kreckel; Vollinga and Weinzierl]

(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

● 85 Master Integrals



Reduze2 - von Manteuffel, Studerus

2 MI's 3 MI's 4 MI's 5 MI's

taken from Pierpaolo's slides at Amplitudes 2014

(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

- ▶ the f 's obey an ϵ -linear DE system ($x = \frac{s}{m^2}$, $y = \frac{t}{m^2}$)

$$\begin{aligned}\partial_x f(x, y, \epsilon) &= (A_{1,0}(x, y) + \epsilon A_{1,1}(x, y)) f(x, y, \epsilon) \\ \partial_y f(x, y, \epsilon) &= (A_{2,0}(x, y) + \epsilon A_{2,1}(x, y)) f(x, y, \epsilon)\end{aligned}$$

- ▶ After getting rid of $A_{i,0}$'s with Magnus, the g 's obey a canonical DE

$$\begin{aligned}\partial_x g(x, y, \epsilon) &= \epsilon \hat{A}_x(x, y) g(x, y, \epsilon) \\ \partial_y g(x, y, \epsilon) &= \epsilon \hat{A}_y(x, y) g(x, y, \epsilon)\end{aligned}$$

- ▶ which can be cast in a *dlog* form

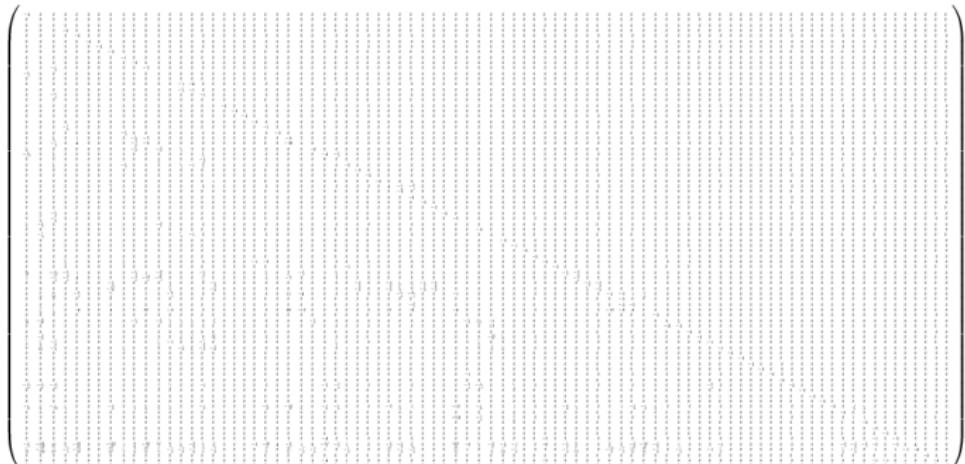
$$dg(x, y, \epsilon) = \epsilon d\mathcal{A}(x, y) g(x, y, \epsilon)$$

- ▶ with *alphabet* $\{x, 1-x, y, 1-y, 1-x-y, x+y\}$

(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

$1/x$

$$\left(\dots \right)$$


(3) Higgs + 1Jet 3-loop ladder

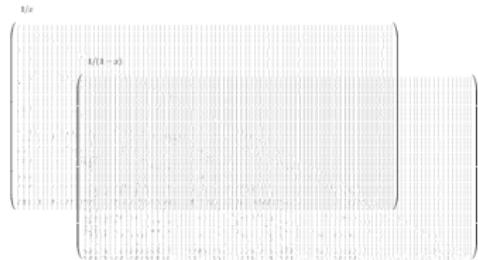
[Mastrolia, Schubert, Yundin, DV... work in progress!]

$1/x$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	- $\frac{3}{2}$	$\frac{1}{2}$	$\frac{9}{4}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	0	0
0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0	0	0	0
0	0	0	$5 - \frac{1}{6}$	$-\frac{11}{4}$	2	0	0	0	0	-1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	$-\frac{3}{2}$	0	0	0	0	0	0	0	0	0	0
0	0	0	12	0	-6	0	0	0	-4	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

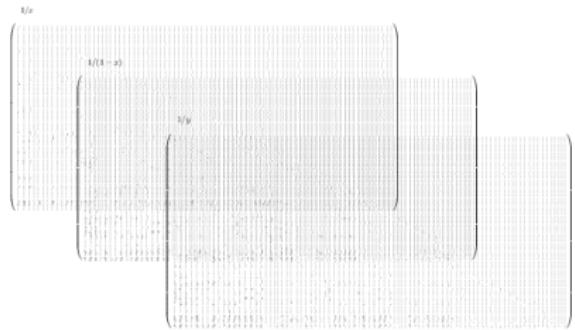
(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

$$\frac{1}{x} \left(\frac{\lambda}{\lambda(1-x)} \left(\frac{\lambda}{\lambda(1-x)} \left(\dots \left(\frac{\lambda}{\lambda(1-x)} \left(\dots \right) \right) \right) \right) \right)$$


(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

$$\frac{1}{x} \left(\frac{1}{1-x} \left(\frac{1}{y} \left(\dots \right) \right) \right)$$


(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

$$\frac{1}{k/x} \left(\frac{1}{k/(1-x)} \left(\frac{1}{k/y} \left(\frac{1}{k/(1-y)} \right) \right) \right)$$

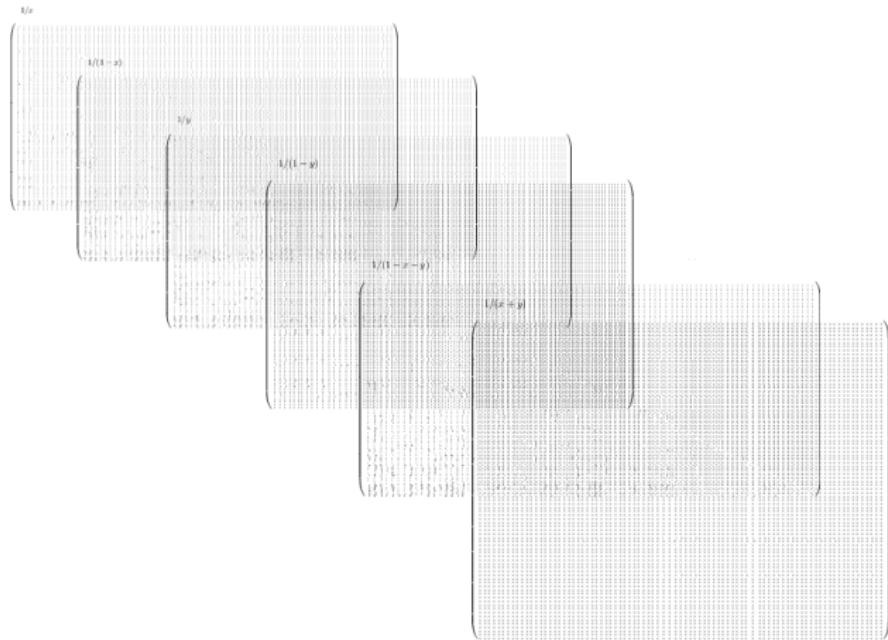
(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]

$$\frac{1}{x} \left(\left(\frac{1}{1-x} \right) \left(\frac{1}{1-y} \right) \left(\frac{1}{1-(x-y)} \right) \right)$$

(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]



(3) Higgs + 1Jet 3-loop ladder

[Mastrolia, Schubert, Yundin, DV... work in progress!]



Conclusions

- ▶ It is more and more urgent to provide precise theoretical predictions, especially in view of LHC physics
- ▶ NLO has reached quite a high degree of automation, so it is time to look at **NNLO and beyond**, with an eye towards *simplicity, compactness and fast numerical evaluation*
- ▶ New ideas stimulated the DE field and brought to fast developments
- ▶ We devised a procedure for obtaining a *canonical basis* of MI's in the case of linear ϵ dependence
- ▶ As a warmup, some results (QED 2-loop vertices, 2-loop massless crossed box) previously known in a “bad” basis were recomputed
- ▶ **New results for the 3-loop Higgs+jet ladder topology are on their way**
- ▶ I just entered the field, but at least I did it with a good timing ;)

The end

Thanks for your attention!

